3. Let $\vec{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Find: **(a)** $2\vec{v} - 3(\vec{u} + \vec{w})$ **(b)** $\|\vec{u}\|$

Solution: (a) $2\vec{v} - 3(\vec{u}+\vec{w}) = 2\begin{pmatrix} 0 \\ 3 \end{pmatrix} - 3\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 6 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ 6 \end{pmatrix}}$

(b) $\|\vec{u}\| = \sqrt{2^2 + (-1)^2} = \boxed{\sqrt{5}}$

4. The vector $\vec{u}$ has initial point $P = (1, 3)$ and terminal point $Q = (-2, 0)$.

(a) Find its position vector.

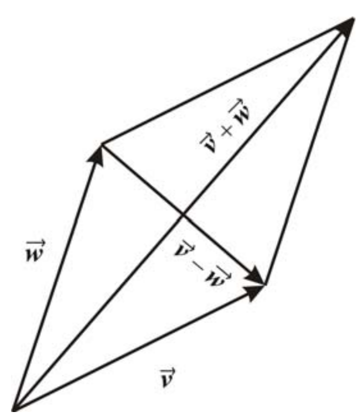
(b) Write $\vec{u}$ in the form $\vec{u} = a\vec{e}_1 + b\vec{e}_2$, where $\vec{e}_1, \vec{e}_2$ are the "basic" vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

Solution: (a) $\vec{u} = \overrightarrow{PQ} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$ **(b)** $\vec{u} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + -3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

6. A parallelogram with sides of equal length is called a **rhombus**. Show that the diagonals of a rhombus are perpendicular.

Solution: The two diagonals are $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$ hence

$$
\begin{aligned}
(\vec{v}-\vec{w})\cdot(\vec{v}+\vec{w}) &= \vec{v}\cdot(\vec{v}+\vec{w}) - \vec{w}\cdot(\vec{v}+\vec{w}) \\
&= \vec{v}\cdot\vec{v} + \underbrace{\vec{v}\cdot\vec{w} - \vec{w}\cdot\vec{v}}_{=0} - \vec{w}\cdot\vec{w} \\
&= \vec{v}\cdot\vec{v} - \vec{w}\cdot\vec{w} \\
&= \|\vec{v}\|^2 - \|\vec{w}\|^2 \qquad [\text{ but } \|\vec{v}\| = \|\vec{w}\| \ ] \\
&= 0
\end{aligned}
$$



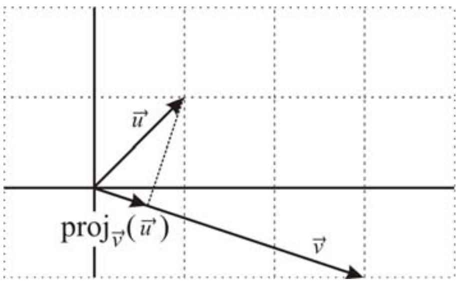
$$(\vec{v}-\vec{w})\cdot(\vec{v}+\vec{w}) = 0 \quad \Rightarrow \quad (\vec{v}-\vec{w}) \perp (\vec{v}+\vec{w})$$

Or (if one wants to use coordinates)

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $\vec{v}+\vec{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$ and $\vec{v}-\vec{w} = \begin{pmatrix} v_1 - w_1 \\ v_2 - w_2 \end{pmatrix}$. Hence

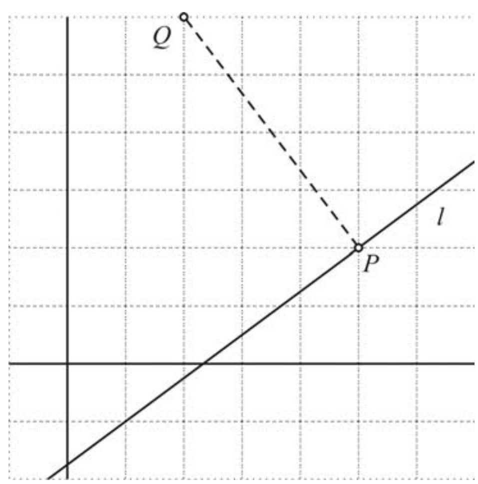
$$(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \end{pmatrix}\cdot\begin{pmatrix} v_1-w_1 \\ v_2-w_2 \end{pmatrix} = (v_1+w_1)(v_1-w_1)+(v_2+w_2)(v_2-w_2)$$
$$= (v_1^2-w_1^2)+(v_2^2-w_2^2)$$
$$= (v_1^2+v_2^2)-(w_1^2+w_2^2)$$
$$= \|\vec{v}\|^2-\|\vec{w}\|^2 = 0 \qquad \therefore \quad (\vec{v}-\vec{w})\perp(\vec{v}+\vec{w})$$

7. Find the orthogonal projection of $\vec{u}=\begin{pmatrix}1\\1\end{pmatrix}$ onto $\vec{v}=\begin{pmatrix}3\\-1\end{pmatrix}$.



$$\text{proj}_{\vec{v}}(\vec{u}) = \|\vec{u}\|\cos(\theta)\frac{\vec{v}}{\|\vec{v}\|} = \|\vec{u}\|\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\frac{\vec{v}}{\|\vec{v}\|}$$
$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$
$$= \frac{3-1}{\left(\sqrt{9+1}\right)^2}\begin{pmatrix}3\\-1\end{pmatrix} = \boxed{\begin{pmatrix}3/5\\-1/5\end{pmatrix} = \begin{pmatrix}0.6\\-0.2\end{pmatrix}}$$

10. Using orthogonal projection find the point P **on** the line $l:\ 3x-4y=7$ closest to the point $Q=(2,6)$



Solutions: To find the orthogonal projection of point Q onto the line l we can consider the line through Q that is perpendicular to l:

$$4x+3y=26$$

and find the intersection P of these two lines:

$$\left.\begin{matrix}4x+3y=26\\3x-4y=7\end{matrix}\right\} \Rightarrow \left.\begin{matrix}16x+12y=104\\9x-12y=21\end{matrix}\right\} \Rightarrow 25x=125 \Rightarrow \boxed{x=5 \quad \therefore \quad y=2}$$

Hence $P=(5,2)$

Or: Let P be a generic point on $l: \ y = \frac{3}{4}x - \frac{7}{4}$: i.e. $P = \left(x, \ \frac{3}{4}x - \frac{7}{4} \right)$

then $\overrightarrow{QP}$ should be perpendicular to the direction vector of the line l : $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

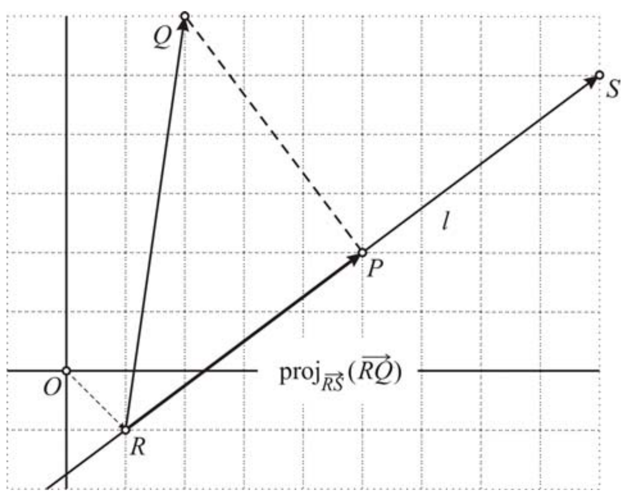
Hence

$$\overrightarrow{QP} = \begin{pmatrix} x-2 \\ \frac{3}{4}x - \frac{7}{4} - 6 \end{pmatrix} \perp \begin{pmatrix} 4 \\ 3 \end{pmatrix} \ \Rightarrow \ \begin{pmatrix} x-2 \\ \frac{3}{4}x - \frac{7}{4} - 6 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 0 \ \Rightarrow \ 4(x-2) + 3\left(\frac{3}{4}x - \frac{7}{4} - 6 \right) = 0$$

$$\Rightarrow \quad x = 5 \quad \therefore \quad y = 2$$

Or (yet another solution)**:** One could also use projection [$\text{proj}_{\vec{v}}(\vec{u})$] as follows:

First pick two points on the line l, say $R = (1, -1)$ and $S = (9, 5)$,



then project $\overrightarrow{RQ} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ onto $\overrightarrow{RS} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$:

$$\overrightarrow{RP} = \text{proj}_{\overrightarrow{RS}}(\overrightarrow{RQ}) = \frac{\overrightarrow{RQ} \cdot \overrightarrow{RS}}{\left\| \overrightarrow{RS} \right\|^2} \overrightarrow{RS} = \frac{\begin{pmatrix} 1 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 6 \end{pmatrix}}{\left(8^2 + 6^2 \right)} \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

$$= \frac{8 + 42}{100} \begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Hence $\overrightarrow{OP} = \overrightarrow{OR} + \overrightarrow{RP} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \therefore \qquad P = (5, 2)$